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Second-Order Boundary Value Problems of Singular Type

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Positive solutions are established for the singular second-order boundary value problem

$$y'' + \phi(t)f(t, y, y') = 0, \qquad 0 < t < 1$$
$$y(0) = y'(1) = 0.$$

Singularities at (i) $y = 0$ and $y' = 0$, (ii) $y = 0$ but not $y' = 0$, and (iii) $y' = 0$ but not $y = 0$ are discussed separately.

1. INTRODUCTION

In the study of nonlinear phenomena, many mathematical models give rise to the singular boundary value problem

$$\begin{aligned} y'' + \phi(t)f(t, y, y') &= 0, \qquad 0 < t < 1 \\ y(0) = y'(1) &= 0, \end{aligned} \tag{1.1}$$

where our nonlinear term f may be singular at

(i) $y = 0$ and $y' = 0$,

(ii) $y = 0$ but not $y' = 0$, or

(iii) $y' = 0$ but not $y = 0$.

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Almost all papers in the literature discuss case (ii), usually when $f(t, u, p)$ is independent of p. We refer the reader to [3, 4, 7, 8] and their references. It is only recently [1] that case (ii) has been examined in its full generality. We note that case (i) and case (iii) have received very little attention in the literature; we mention in particular [5, 6, 9]. The goal of this paper is to attempt to fill this gap in the literature. In particular, we will establish a general existence result for case (i) in Section 3 and case (iii) in Section 2. In addition, in Section 4, we will obtain a new result for case (ii). It is worth remarking here that one could use the ideas of this paper to discuss other boundary value problems, for example,

$$\frac{1}{p}(py')' + \phi(t)f(t, y, py') = 0, \qquad 0 < t < 1$$
$$\lim_{t \to 0^+} p(t)y'(t) = y(1) = 0.$$

For the remainder of this section we gather together two results that will be used throughout this paper. Suppose $y \in C^1[0,1] \cap C^2(0,1)$ satisfies

$$\begin{aligned} -y'' &> 0 \qquad \text{on } (0,1) \\ y(0) &= 0 \\ y'(1) &= a \geq 0. \end{aligned}$$

In [2] we showed that

$$y(t) \geq ty(1) = t \sup_{t \in [0,1]} |y(t)| \tag{1.2}$$

for $t \in [0,1]$. Next consider

$$\begin{aligned} y'' + \phi(t)F(t, y, y') &= 0, \qquad 0 < t < 1 \\ y(0) &= a \geq 0 \\ y'(1) &= b \geq 0. \end{aligned} \tag{1.3}$$

THEOREM 1.1. [1]. *Suppose*

$$F : [0,1] \times \mathbf{R}^2 \to \mathbf{R} \qquad \textit{is continuous} \tag{1.4}$$

and

$$\phi \in C(0,1) \qquad \textit{with } \phi > 0 \textit{ on } (0,1) \quad \textit{and} \quad \phi \in L^1[0,1] \tag{1.5}$$

are satisfied. In addition, assume there are constants $M_0 > a + b$ and $M_1 > b$, independent of λ, with

$$|y|_1 = \max\left\{\frac{|y|_0}{M_0}, \frac{|y'|_0}{M_1}\right\} \neq 1$$

for any solution $y \in C^1[0,1] \cap C^2(0,1)$ *to*

$$\begin{aligned} y'' + \lambda\phi(t)F(t,y,y') &= 0, \qquad 0 < t < 1 \\ y(0) &= a \\ y'(1) &= b \end{aligned} \tag{1.6}$$

for each $\lambda \in (0,1)$; here $|u|_0 = \sup_{(0,1)}|u(t)|$. *Then* (1.3) *has a solution* $y \in C^1[0,1] \cap C^2(0,1)$ with $|y|_1 \le 1$.

2. SINGULARITIES AT $y' = 0$ BUT NOT AT $y = 0$

In this section we discuss (1.1). Our nonlinearity f may be singular at $y' = 0$, but is *not* singular at $y = 0$. Throughout this section we will assume the following conditions hold:

$$\phi \in C(0,1) \quad \text{with } \phi > 0 \text{ on } (0,1) \quad \text{and } \phi \in L^1[0,1] \tag{2.1}$$

$$\begin{aligned} &f : [0,1] \times [0,\infty) \times (0,\infty) \to [0,\infty) \qquad \text{is continuous with} \\ &f(t,u,p) > 0 \qquad \text{for } (t,u,p) \in [0,1] \times (0,\infty) \times (0,\infty) \end{aligned} \tag{2.2}$$

$$\begin{aligned} &f(t,u,p) \le h(u)[g(p) + r(p)] \\ &\qquad\qquad \text{on } [0,1] \times (0,\infty) \times (0,\infty) \text{ with} \\ &g > 0 \qquad \text{continuous and nonincreasing on } (0,\infty), \text{ and} \\ &h \ge 0,\ r \ge 0 \qquad \text{continuous and nondecreasing on } [0,\infty) \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\sup_{c \in (0,\infty)} \frac{c}{I^{-1}\left(h(c)\int_0^1 \phi(s)\,ds\right)} > 1 \qquad \text{where} \\ &I(z) = \int_0^z \frac{du}{g(u) + r(u)} \qquad \text{for } z > 0 \end{aligned} \tag{2.4}$$

$$I(\infty) = \infty \tag{2.5}$$

for constants $H > 0$, $L > 0$ there exists a function $\psi_{H,L}$ continuous on $[0,1]$ and positive on $(0,1)$, and a constant γ, $0 \le \gamma < 1$, with $f(t,u,p) \ge \psi_{H,L}(t)u^{\gamma}$ on $[0,1] \times [0,H] \times (0,L]$ (2.6)

and

$$\int_0^1 \phi(t)\, g\left(k_0 \int_t^1 s^{\gamma}\psi_{H,L}(s)\,\phi(s)\,ds\right) dt < \infty \qquad \text{for any constant } k_0 > 0. \tag{2.7}$$

THEOREM 2.1. *Suppose* (2.1)–(2.7) *hold. Then* (1.1) *has a solution* $y \in C^1[0,1] \cap C^2(0,1)$ *with* $y > 0$ *on* $(0,1]$.

Proof. Choose $M > 0$ with

$$\frac{M}{I^{-1}\left(h(M)\int_0^1 \phi(s)\,ds\right)} > 1. \tag{2.8}$$

Next choose $\epsilon > 0$ and $\epsilon < M$ with

$$\frac{M}{I^{-1}\left(h(M)\int_0^1 \phi(s)\,ds + I(\epsilon)\right)} > 1. \tag{2.9}$$

Let $n_0 \in \{1, 2, \ldots\}$ be chosen so that $1/n_0 < \epsilon$, and let $N_0 = \{n_0, n_0 + 1, \ldots\}$. We first show that

$$\begin{gathered} y'' + \phi(t) f^*(t, y, y') = 0, \quad 0 < t < 1 \\ y(0) = 0, \quad y'(1) = \frac{1}{m} \end{gathered} \tag*{$(2.10)^m$}$$

has a solution for each $m \in N_0$; here

$$f^*(t,u,p) = \begin{cases} f(t,u,p), & u \geq 0, \quad p \geq \dfrac{1}{m} \\ f\left(t,u,\dfrac{1}{m}\right), & u \geq 0, \quad p < \dfrac{1}{m} \\ f(t,0,p), & u < 0, \quad p \geq \dfrac{1}{m} \\ f\left(t,0,\dfrac{1}{m}\right), & u < 0, \quad p < \dfrac{1}{m} \end{cases}.$$

To show $(2.10)^m$ has a solution, we consider the family of problems

$$\begin{gathered} y'' + \lambda\phi(t) f^*(t, y, y') = 0, \quad 0 < t < 1 \\ y(0) = 0, \quad y'(1) = \frac{1}{m}, \qquad m \in N_0 \end{gathered} \tag*{$(2.11)_\lambda^m$}$$

for $0 < \lambda < 1$. Let $y \in C^1[0,1] \cap C^2(0,1)$ be any solution of $(2.11)_\lambda^m$. The differential equation and (2.2) immediately imply that $y'' \leq 0$ on $(0,1)$, $y' \geq (1/m)$ on $[0,1]$, and $y \geq (1/m)$ on $[0,1]$. Also from (2.3) we have

$$\frac{-y''(t)}{g(y'(t)) + r(y'(t))} \leq h(y(t))\phi(t) \leq h(y(1))\phi(t) \qquad \text{for } t \in (0,1).$$

Integration from t to 1 yields

$$I(y'(t)) - I\left(\frac{1}{m}\right) \leq h(y(1)) \int_0^1 \phi(s)\, ds,$$

and so

$$y'(t) \leq I^{-1}\left(h(y(1)) \int_0^1 \phi(s)\, ds + I(\epsilon)\right) \qquad \text{for } t \in [0,1]. \quad (2.12)$$

Now integrate from 0 to 1 to obtain

$$\frac{y(1)}{I^{-1}\left(h(y(1)) \int_0^1 \phi(s)\, ds + I(\epsilon)\right)} \leq 1. \tag{2.13}$$

Now (2.9) together with (2.13) implies

$$|y|_0 = y(1) \neq M. \tag{2.14}$$

Next notice any solution y of $(2.11)_\lambda^m$ that satisfies $0 \leq y(t) \leq M$ for $t \in [0,1]$ also satisfies (see (2.12))

$$\frac{1}{m} \leq y'(t) < I^{-1}\left(h(M) \int_0^1 \phi(s)\, ds + I(\epsilon)\right) + 1 \equiv M_1 \qquad \text{for } t \in [0,1]. \tag{2.15}$$

Let $M_0 = M$ and $M_1 = M_1$ in Theorem 1.1. Notice from (2.14) and (2.15) that

$$|y|_1 = \max\left\{\frac{|y|_0}{M_0}, \frac{|y'|_0}{M_1}\right\} \neq 1. \tag{2.16}$$

To see this, notice whether $|y|_1 = 1$; then because of (2.14), $|y'|_0 = M_1$. But then $|y|_0 < M_0$. This together with (2.15) implies $|y'|_0 < M_1$. We have a contradiction, so (2.16) is true.

Thus Theorem 1.1 implies $(2.10)^m$ has a solution y_m with $|y_m|_1 < 1$. In fact,

$$0 \leq y_m(t) < M \quad \text{and} \quad \frac{1}{m} \leq y_m'(t) < M_1 \qquad \text{for } t \in [0,1], \quad (2.17)$$

and y_m satisfies

$$y'' + \phi(t) f(t, y, y') = 0, \qquad 0 < t < 1$$
$$y(0) = 0, \quad y'(1) = \frac{1}{m}.$$

Next notice that (2.6) guarantees the existence of a function $\psi_{M, M_1}(t)$ continuous on $[0, 1]$ and positive on $(0, 1)$ and a constant γ, $0 \le \gamma < 1$, with $f(t, y_m(t), y_m'(t)) \ge \psi_{M, M_1}(t)[y_m(t)]^\gamma$ for $(t, y_m(t), y_m'(t)) \in [0, 1] \times [0, M] \times (0, M_1]$. We claim

$$y_m'(t) \ge \int_t^1 s^\gamma \psi_{M, M_1}(s) \phi(s) \left(\int_0^1 x^{\gamma+1} \psi_{M, M_1}(x) \phi(x)\, dx \right)^{\gamma/(1-\gamma)} ds \quad (2.18)$$

for $t \in [0, 1]$.

Remark 2.1. Notice that (2.18) is immediate if $\gamma = 0$.

To see (2.18), first notice

$$y_m(t) \ge \int_0^t s\phi(s) f(s, y_m(s), y_m'(s))\, ds + t \int_t^1 \phi(s) f(s, y_m(s), y_m'(s))\, ds,$$

and so,

$$y_m(1) \ge \int_0^1 s\phi(s) \psi_{M, M_1}(s) [y_m(s)]^\gamma\, ds.$$

This together with (1.2) gives

$$y_m(1) \ge \int_0^1 s\phi(s) \psi_{M, M_1}(s) [s y_m(1)]^\gamma\, ds,$$

and so,

$$y_m(1) \ge \left(\int_0^1 s^{\gamma+1} \psi_{M, M_1}(s) \phi(s)\, ds \right)^{1/(1-\gamma)} \equiv a_0. \qquad (2.19)$$

Now

$$y_m'(t) \ge \int_t^1 \phi(s) \psi_{M, M_1}(s) [y_m(s)]^\gamma\, ds \ge \int_t^1 \phi(s) \psi_{M, M_1}(s) [s y_m(1)]^\gamma\, ds,$$

and this together with (2.19) implies

$$y_m'(t) \ge a_0^\gamma \int_t^1 s^\gamma \phi(s) \psi_{M, M_1}(s)\, ds,$$

so (2.18) is true.

Next we show

$$\begin{aligned}&\{y_m^{(j)}\}_{m\in N_0} \text{ is a bounded, equicontinuous}\\&\text{family on } [0,1] \text{ for each } j=0,1.\end{aligned} \tag{2.20}$$

We need only check equicontinuity since (2.17) holds. Of course for $t \in (0,1)$ we have

$$\begin{aligned}0 \le -y_m''(t) &\le h(M)\left[g(y_m'(t)) + r(M_1)\right]\phi(t)\\&\le h(M)\left[g\left(a_0\int_t^1 s^\gamma \phi(s)\,\psi_{M,M_1}(s)\,ds\right) + r(M_1)\right]\phi(t).\end{aligned}$$

Now (2.20) is immediate from the above, (2.7) and (2.17).

The Arzela–Ascoli Theorem guarantees the existence of a subsequence N of N_0 and a function $y \in C^1[0,1]$ with $y_m^{(j)}$ converging uniformly on $[0,1]$ to $y^{(j)}$ as $m \to \infty$ through N; here $j=0,1$. Also $y(0)=0=y'(1)$. In addition, since $y_m'(t) \ge a_0^\gamma \int_t^1 s^\gamma \phi(s)\psi_{M,M_1}(s)\,ds$ for $t\in[0,1]$, we have

$$y'(t) \ge a_0^\gamma \int_t^1 s^\gamma \phi(s)\,\psi_{M,M_1}(s)\,ds \qquad \text{for } t\in[0,1],$$

and so $y' > 0$ on $[0,1)$ and $y>0$ on $(0,1]$. Now y_m, $m \in N$, satisfies

$$y_m'(t) = y_m'(0) - \int_0^t \phi(s) f(s, y_m(s), y_m'(s))\,ds \qquad \text{for } t \in [0,1]. \tag{2.21}$$

Fix $t\in[0,1)$. Let $m\to\infty$ through N in (2.21) to obtain

$$y'(t) = y'(0) - \int_0^t \phi(s) f(s, y(s), y'(s))\,ds \qquad \text{for } t \in [0,1]. \tag{2.22}$$

From (2.22) we deduce immediately that $y \in C^2(0,1)$ and $y''(t) + \phi(t)f(t,y(t),y'(t)) = 0$ for $t\in(0,1)$. ∎

EXAMPLE 2.1. Consider the boundary value problem

$$\begin{aligned}y'' + \mu(y')^{-\alpha}\left[y^\beta + 1\right] = 0, &\qquad 0<t<1\\ y(0) = y'(1) = 0,&\end{aligned} \tag{2.23}$$

with $0<\alpha<1$, $\beta \ge 0$, and $\mu > 0$. If

$$\mu < (\alpha+1)\left(\sup_{c\in(0,\infty)} \frac{c}{\left[c^\beta + 1\right]^{1/(\alpha+1)}}\right)^{\alpha+1}, \tag{2.24}$$

then (2.23) has a solution $y \in C^1[0,1] \cap C^2(0,1)$ with $y>0$ on $(0,1]$.

Remark 2.2. If $\beta < \alpha + 1$, then (2.24) is satisfied for all $\mu > 0$.

To see that (2.23) has a solution, we will apply Theorem 2.1 with $\phi = 1$, $g(u) = u^{-\alpha}$, $r = 0$, and $h(u) = \mu[u^{\beta} + 1]$. Clearly, (2.1), (2.2), (2.3), (2.6) (with $\psi_{H,L} = L^{-\alpha}$ and $\gamma = 0$), and (2.7) (since $0 < \alpha < 1$) are satisfied. Next notice that $I(z) = z^{\alpha+1}/(\alpha + 1)$, so (2.5) holds. Also,

$$\sup_{c \in (0,\infty)} \frac{c}{I^{-1}\left(h(c)\int_0^1 \phi(s)\,ds\right)} = \sup_{c \in (0,\infty)} \frac{c}{(\alpha+1)^{1/(\alpha+1)}\mu^{1/(\alpha+1)}[c^{\beta}+1]^{1/(\alpha+1)}},$$

so (2.24) guarantees that (2.4) holds. Theorem 2.1 now establishes the result.

EXAMPLE 2.2. Consider the boundary value problem

$$\begin{gathered} y'' + \mu(y')^{-\alpha} y^{\beta} = 0, \qquad 0 < t < 1 \\ y(0) = y'(1) = 0, \end{gathered} \tag{2.25}$$

with $0 < \alpha < 1, 0 \le \beta < 1$, and $\mu > 0$. Then (2.25) has a solution $y \in C^1[0,1] \cap C^2(0,1)$ with $y > 0$ on $(0,1]$.

We will apply Theorem 2.1 with $\phi = 1$, $g(u) = u^{-\alpha}$, and $h(u) = \mu u^{\beta}$. Clearly, (2.1), (2.2), (2.3), (2.5), (2.6) (with $\psi_{H,L} = L^{-\alpha}$ and $\gamma = \beta$), and (2.7) (since $0 < \alpha < 1$) hold. Finally, (2.4) is satisfied, since

$$\sup_{c \in (0,\infty)} \frac{c}{I^{-1}\left(h(c)\int_0^1 \phi(s)\,ds\right)} = \sup_{c \in (0,\infty)} \frac{c}{(\alpha+1)^{1/(\alpha+1)}\mu^{1/(\alpha+1)}c^{\beta/(\alpha+1)}} = \infty.$$

Theorem 2.1 now guarantees the result.

3. SINGULARITIES AT $y' = 0$ AND $y = 0$

In this section our nonlinearity f may be singular at $y' = 0$ and $y = 0$. Throughout this section we will assume that the following conditions hold:

$$\phi \in C[0,1] \qquad \text{with } \phi > 0 \text{ on } (0,1) \tag{3.1}$$

$$f : [0,1] \times (0,\infty) \times (0,\infty) \to (0,\infty) \qquad \text{is continuous} \tag{3.2}$$

$$\left.\begin{array}{l} f(t,u,p) \le [h(u)+w(u)][g(p)+r(p)] \\ \qquad\qquad \text{on } [0,1]\times(0,\infty)\times(0,\infty) \text{ with} \\ g>0, w>0 \quad \text{continuous and nonincreasing on } (0,\infty), \text{ and} \\ h\ge 0, r\ge 0 \quad \text{continuous and nondecreasing on } [0,\infty) \end{array}\right. \tag{3.3}$$

$$\sup_{c\in(0,\infty)} \frac{c}{I^{-1}\left(ch(c)|\phi|_0 + |\phi|_0 \int_0^c w(x)\,dx\right)} > 1$$

$$\text{where } I(z) = \int_0^z \frac{u\,du}{g(u)+r(u)}, \quad z>0 \tag{3.4}$$

$$I(\infty)=\infty \quad \text{and} \quad \int_0^a w(x)\,dx < \infty \qquad \text{for any } a>0 \tag{3.5}$$

$$\left.\begin{array}{l} \text{for constants } H>0, L>0 \text{ there exists a function } \psi_{H,L} \\ \text{continuous on } [0,1] \text{ and positive on } (0,1), \text{ with } f(t,u,p) \ge \\ \psi_{H,L}(t) \text{ on } [0,1]\times(0,H]\times(0,L] \end{array}\right. \tag{3.6}$$

and

$$\int_0^1 \phi(t)w(k_0 t)\,dt < \infty \quad \text{and} \quad \int_0^1 \phi(t)g\left(\int_t^1 \psi_{H,L}(s)\phi(s)\,ds\right)dt < \infty$$
$$\text{for any constant } k_0>0. \tag{3.7}$$

THEOREM 3.1. *Suppose* (3.1)–(3.7) *hold. Then* (1.1) *has a solution* $y \in C^1[0,1] \cap C^2(0,1)$ *with* $y>0$ *on* $(0,1]$.

Proof. Choose $M>0$, and then $\epsilon>0$ with $\epsilon < M/2$ and with

$$\frac{M}{I^{-1}\left(Mh(M)|\phi|_0 + |\phi|_0 \int_0^M w(x)\,dx + I(\epsilon)\right)+\epsilon} > 1. \tag{3.8}$$

Choose $n_0 \in \{1,2,\dots\}$ with $1/n_0 < \epsilon$, and let $N_0 = \{n_0, n_0+1,\dots\}$. We first show that

$$\left.\begin{array}{l} y'' + \phi(t)f^{**}(t,y,y') = 0, \qquad 0<t<1 \\ y(0) = y'(1) = \dfrac{1}{m} \end{array}\right. \tag*{$(3.9)^m$}$$

has a solution for each $m \in N_0$; here

$$f^{**}(t,u,p) = \begin{cases} f(t,u,p), & u \geq \dfrac{1}{m}, \quad p \geq \dfrac{1}{m} \\ f\left(t,u,\dfrac{1}{m}\right), & u \geq \dfrac{1}{m}, \quad p < \dfrac{1}{m} \\ f\left(t,\dfrac{1}{m},p\right), & u < \dfrac{1}{m}, \quad p \geq \dfrac{1}{m} \\ f\left(t,\dfrac{1}{m},\dfrac{1}{m}\right), & u < \dfrac{1}{m}, \quad p < \dfrac{1}{m} \end{cases}.$$

Consider the family of problems

$$\begin{gathered} y'' + \lambda\phi(t) f^{**}(t,y,y') = 0, \qquad 0 < t < 1 \\ y(0) = y'(1) = \frac{1}{m}, \qquad m \in N_0 \end{gathered} \tag{3.10$^m_\lambda$}$$

for $0 < \lambda < 1$. Let $y \in C^1[0,1] \cap C^2(0,1)$ be any solution of $(3.10)^m_\lambda$. It is immediate that $y' \geq 1/m$ and $y \geq 1/m$ on $[0,1]$. In addition we have

$$\frac{-y''(t)}{g(y'(t)) + r(y'(t))} \leq \left[h(y(t)) + w(y(t))\right]\phi(t) \qquad \text{for } t \in (0,1), \tag{3.11}$$

and so,

$$\frac{-y'(t)y''(t)}{g(y'(t)) + r(y'(t))} \leq |\phi|_0\left[h(y(1)) + w(y(t))\right]y'(t) \qquad \text{for } t \in (0,1).$$

Integration from t to 1 yields

$$I(y'(t)) - I\left(\frac{1}{m}\right) \leq |\phi|_0 h(y(1))y(1) + |\phi|_0\int_{y(t)}^{y(1)} w(x)\,dx,$$

and so,

$$y'(t) \leq I^{-1}\left(|\phi|_0 h(y(1))y(1) + |\phi|_0\int_0^{y(1)} w(x)\,dx + I(\epsilon)\right) \qquad \text{for } t \in [0,1]. \tag{3.12}$$

Now integrate from 0 to 1 to obtain

$$y(1) \leq \frac{1}{m} + I^{-1}\left(|\phi|_0 h(y(1))y(1) + |\phi|_0 \int_0^{y(1)} w(x)\,dx + I(\epsilon)\right),$$

and so,

$$\frac{y(1)}{\epsilon + I^{-1}\left(|\phi|_0 h(y(1))y(1) + |\phi|_0 \int_0^{y(1)} w(x)\,dx + I(\epsilon)\right)} \leq 1. \quad (3.13)$$

Now (3.8) together with (3.13) implies

$$|y|_0 = y(1) \neq M. \quad (3.14)$$

Next notice any solution y of $(3.10)_\lambda^m$ that satisfies $1/m \leq y(t) \leq M$ for $t \in [0,1]$ also satisfies (see (3.12))

$$\frac{1}{m} \leq y'(t) < I^{-1}\left(|\phi|_0 h(M)M + |\phi|_0 \int_0^M w(x)\,dx + I(\epsilon)\right) + 1 \equiv M_1$$
$$\text{for } t \in [0,1]. \quad (3.15)$$

Let $M_0 = M$ and $M_1 = M_1$ in Theorem 1.1. Notice from (3.14) and (3.15) that

$$|y|_1 = \max\left\{\frac{|y|_0}{M_0}, \frac{|y'|_0}{M_1}\right\} \neq 1.$$

Thus Theorem 1.1 implies $(3.9)^m$ has a solution y_m with $|y_m|_1 < 1$. In fact,

$$\frac{1}{m} \leq y_m(t) < M \quad \text{and} \quad \frac{1}{m} \leq y_m'(t) < M_1 \qquad \text{for } t \in [0,1], \quad (3.16)$$

and y_m satisfies

$$y'' + \phi(t)f(t,y,y') = 0, \qquad 0 < t < 1$$
$$y(0) = y'(1) = \frac{1}{m}.$$

Next notice (3.6) guarantees the existence of a function $\psi_{M,M_1}(t)$ continuous on $[0,1]$ and positive on $(0,1)$ with $f(t, y_m(t), y_m'(t)) \geq \psi_{M,M_1}(t)$ for $(t, y_m(t), y_m'(t)) \in [0,1] \times (0,M] \times (0,M_1]$. Of course we have immediately that

$$y_m'(t) \geq \int_t^1 \phi(s)\psi_{M,M_1}(s)\,ds \qquad \text{for } t \in [0,1]. \quad (3.17)$$

Also,

$$y_m(t) \geq \int_0^t s\phi(s)\psi_{M,M_1}(s)\,ds + t\int_t^1 \phi(s)\psi_{M,M_1}(s)\,ds,$$

and so

$$y_m(t) \geq t\Omega_{M,M_1}(t) \qquad \text{for } t \in [0,1]; \tag{3.18}$$

here

$$\Omega_{M,M_1}(t) = \frac{1}{t}\int_0^t s\phi(s)\psi_{M,M_1}(s)\,ds + \int_t^1 \phi(s)\psi_{M,M_1}(s)\,ds.$$

Now since $\lim_{t\to 0+}\Omega_{M,M_1}(t) = \int_0^1 \phi(s)\psi_{M,M_1}(s)\,ds$, Ω_{M,M_1} extends to a continuous function on $[0,1]$. Consequently there exists a $k_0 > 0$ with $\Omega_{M,M_1}(t) \geq k_0 > 0$ for $t \in [0,1]$. This together with (3.18) implies

$$y_m(t) \geq k_0 t \qquad \text{for } t \in [0,1]. \tag{3.19}$$

Also for $t \in (0,1)$ we have from (3.17) and (3.19) that

$$\begin{aligned} 0 \leq -y_m''(t) &\leq [h(M) + w(y_m(t))][g(y_m'(t)) + r(M_1)]\phi(t) \\ &\leq [h(M) + w(k_0 t)]\left[g\left(\int_t^1 \phi(s)\psi_{M,M_1}(s)\,ds\right) + r(M_1)\right]\phi(t). \end{aligned}$$

Consequently,

$$\begin{gathered} \{y_m^{(j)}\}_{m\in N_0} \text{ is a bounded, equicontinuous} \\ \text{family on } [0,1] \text{ for each } j = 0,1. \end{gathered} \tag{3.20}$$

The Arzela–Ascoli Theorem guarantees the existence of a subsequence N of N_0 and a function $y \in C^1[0,1]$ with $y_m^{(j)}$ converging uniformly on $[0,1]$ to $y^{(j)}$ as $m \to \infty$ through N; here $j = 0,1$. Also $y(0) = 0 = y'(1)$ with $y(t) \geq k_0 t$ for $t \in [0,1]$, and $y'(t) \geq \int_t^1 \phi(s)\psi_{M,M_1}(s)\,ds$ for $t \in [0,1]$. In addition, $y_m, m \in N$, satisfies (2.21). Let $m \to \infty$ through N to deduce that $y'' + \phi(t)f(t,y,y') = 0$ for $t \in (0,1)$. ∎

Remark 3.1. It is easy to relax (3.1) as follows: $\phi \in C(0,1)$ with $\phi > 0$ on $(0,1)$ and $\phi \in L^p[0,1]$ for some constant p, $1 \leq p \leq \infty$. Now of course we need to adjust appropriately the other assumptions. Essentially the same reasoning as in Theorem 3.1 will establish the result. The only major change is that (3.11) is multiplied by $(y')^{1/q}$ instead of y'; here q is the conjugate to p.

EXAMPLE 3.1. Consider the boundary value problem

$$\begin{gathered} y'' + \mu(y')^{-\alpha}\left[y^{-\beta} + \eta_0 y^{\gamma} + \eta_1\right] = 0, \qquad 0 < t < 1 \\ y(0) = y'(1) = 0 \end{gathered} \tag{3.21}$$

with $0 < \alpha < 1$, $0 < \beta < 1$, $\eta_0 \geq 0$, $\eta_1 \geq 0$, $\gamma \geq 0$, and $\mu > 0$. If

$$\mu < (\alpha + 2)\left(\sup_{c\in(0,\infty)} \frac{c}{\left[\eta_0 c^{\gamma+1} + \eta_1 c + c^{1-\beta}/(1-\beta)\right]^{1/(\alpha+2)}}\right)^{\alpha+2}, \tag{3.22}$$

then (3.21) has a solution $y \in C^1[0,1] \cap C^2(0,1)$ with $y > 0$ on $(0,1]$.

We will apply Theorem 3.1 with $\phi = \mu$, $g(u) = u^{-\alpha}$, $r = 0$, $w(u) = u^{-\beta}$, and $h(u) = \eta_0 u^{\gamma} + \eta_1$. Clearly, (3.1), (3.2), (3.3), (3.5) (since $0 < \beta < 1$), (3.6) (with $\psi_{H,L} = H^{-\beta}L^{-\alpha}$), and (3.7) (since $0 < \alpha < 1$ and $0 < \beta < 1$) are satisfied. Also,

$$\begin{aligned} &\sup_{c\in(0,\infty)} \frac{c}{I^{-1}\left(ch(c)|\phi|_0 + |\phi|_0 \int_0^c w(x)\,dx\right)} \\ &\quad = \sup_{c\in(0,\infty)} \frac{c}{(\alpha+2)^{1/(\alpha+2)}\mu^{1/(\alpha+2)} \times\left[\eta_0 c^{\gamma+1} + \eta_1 c + c^{1-\beta}/(1-\beta)\right]^{1/(\alpha+2)}}, \end{aligned}$$

so (3.22) guarantees that (3.4) holds. Theorem 3.1 now establishes the result.

4. SINGULARITIES AT $y = 0$ BUT NOT AT $y' = 0$

This section discusses the case when our nonlinearity f may be singular at $y = 0$ but not at $y' = 0$. Throughout this section we will assume that the following conditions hold:

$$\phi \in C[0,1] \qquad \text{with } \phi > 0 \text{ on } (0,1) \tag{4.1}$$

$$\begin{gathered} f:[0,1]\times(0,\infty)\times[0,\infty) \to [0,\infty) \text{ is continuous with} \\ f(t,u,p) > 0 \text{ for } (t,u,p) \in [0,1]\times(0,\infty)\times(0,\infty) \end{gathered} \tag{4.2}$$

$$\begin{aligned} &f(t,u,p) \le [h(u) + w(u)]r(p) \\ &\qquad\qquad \text{on } [0,1] \times (0,\infty) \times (0,\infty) \text{ with} \\ &w > 0 \quad \text{continuous and nonincreasing on } (0,\infty), \text{ and} \\ &h \ge 0, r \ge 0 \quad \text{continuous and nondecreasing on } [0,\infty) \end{aligned} \tag{4.3}$$

$$\sup_{c \in (0,\infty)} \frac{c}{I^{-1}\left(ch(c)|\phi|_0 + |\phi|_0 \int_0^c w(x)\,dx\right)} > 1$$

$$\text{where } I(z) = \int_0^z \frac{u\,du}{r(u)}, \quad z > 0 \tag{4.4}$$

$$I(\infty) = \infty \quad \text{and} \quad \int_0^a w(x)\,dx < \infty \qquad \text{for any } a > 0 \tag{4.5}$$

$$\begin{aligned} &\text{for constants } H > 0,\ L > 0 \text{ there exists a function } \psi_{H,L} \\ &\text{continuous on } [0,1] \text{ and positive on } (0,1), \text{ and a constant} \\ &\gamma,\ 0 \le \gamma < 1, \text{ with } f(t,u,p) \ge \psi_{H,L}(t)p^\gamma \text{ on} \\ &[0,1] \times (0,H] \times [0,L] \end{aligned} \tag{4.6}$$

and

$$\int_0^1 \phi(t) w(k_0 t)\,dt < \infty \qquad \text{for any constant } k_0 > 0. \tag{4.7}$$

THEOREM 4.1. *Suppose* (4.1)–(4.7) *hold. Then* (1.1) *has a solution* $y \in C^1[0,1] \cap C^2(0,1)$ *with* $y > 0$ *on* $(0,1]$.

Proof. Choose $M > 0$, and then $\epsilon > 0$ with $\epsilon < M/2$ and with

$$\frac{M}{I^{-1}\left(Mh(M)|\phi|_0 + |\phi|_0 \int_0^M w(x)\,dx + I(\epsilon)\right) + \epsilon} > 1. \tag{4.8}$$

Choose $n_0 \in \{1, 2, \dots,\}$ with $1/n_0 < \epsilon$ and let $N_0 = \{n_0, n_0 + 1, \dots,\}$. We first show that

$$\begin{aligned} y'' + \phi(t) f^{**}(t, y, y') &= 0, \qquad 0 < t < 1 \\ y(0) = y'(1) &= \frac{1}{m} \end{aligned} \tag*{$(4.9)^m$}$$

has a solution for each $m \in N_0$; here f^{**} is as in Theorem 3.1. Consider the family of problems

$$\begin{aligned} y'' + \lambda\phi(t) f^{**}(t, y, y') &= 0, \qquad 0 < t < 1 \\ y(0) = y'(1) &= \frac{1}{m}, \qquad m \in N_0 \end{aligned} \tag*{$(4.10)^m_\lambda$}$$

for $0 < \lambda < 1$. Let $y \in C^1[0,1] \cap C^2(0,1)$ by any solution of $(4.10)_\lambda^m$. Then $y' \geq 1/m$ and $y \geq 1/m$ on $[0,1]$. Also,

$$\frac{-y'(t)y''(t)}{r(y'(t))} \leq |\phi|_0 [h(y(1)) + w(y(t))] y'(t) \qquad \text{for } t \in (0,1).$$

Essentially the same reasoning as in Theorem 3.1 establishes

$$y'(t) \leq I^{-1}\left(|\phi|_0 h(y(1)) y(1) + |\phi|_0 \int_0^{y(1)} w(x)\,dx + I(\epsilon)\right) \qquad \text{for } t \in [0,1] \quad (4.11)$$

and

$$\frac{y(1)}{\epsilon + I^{-1}\left(|\phi|_0 h(y(1)) y(1) + |\phi|_0 \int_0^{y(1)} w(x)\,dx + I(\epsilon)\right)} \leq 1. \quad (4.12)$$

Now (4.8) together with (4.12) implies

$$|y|_0 = y(1) \neq M. \tag{4.13}$$

Also notice any solution y of $(4.10)_\lambda^{\mathrm{m}}$ that satisfies $1/m \leq y(t) \leq M$ for $t \in [0,1]$ also satisfies

$$\frac{1}{m} \leq y'(t) < I^{-1}\left(|\phi|_0 h(M) M + |\phi|_0 \int_0^M w(x)\,dx + I(\epsilon)\right) + 1 \equiv M_1 \qquad \text{for } t \in [0,1]. \quad (4.14)$$

Let $M_0 = M$ and $M_1 = M_1$ in Theorem 1.1. Now

$$|y|_1 = \max\left\{\frac{|y|_0}{M_0}, \frac{|y'|_0}{M_1}\right\} \neq 1.$$

Thus Theorem 1.1 implies $(4.9)^m$ has a solution y_m with $|y_m|_1 < 1$. In fact,

$$\frac{1}{m} \leq y_m(t) < M \quad \text{and} \quad \frac{1}{m} \leq y_m'(t) < M_1 \qquad \text{for } t \in [0,1]. \quad (4.15)$$

Furthermore, (4.6) guarantees the existence of a function $\psi_{M, M_1}(t)$ continuous on $[0,1]$ and positive on $(0,1)$, and a constant γ, $0 \leq \gamma < 1$, with

$f(t, y_m(t), y'_m(t)) \geq \psi_{M, M_1}(t)[y'_m(t)]^\gamma$ for $(t, y_m(t), y'_m(t)) \in [0, 1] \times (0, M] \times [0, M_1]$. This implies

$$y'_m(t) \geq \left((1-\gamma)\int_t^1 \phi(s)\psi_{M, M_1}(s)\, ds\right)^{1/(1-\gamma)} \qquad \text{for } t \in [0, 1], \tag{4.16}$$

and so,

$$y_m(t) \geq \int_0^t \left((1-\gamma)\int_x^1 \phi(s)\psi_{M, M_1}(s)\, ds\right)^{1/(1-\gamma)} dx \qquad \text{for } t \in [0, 1]. \tag{4.17}$$

Consequently,

$$y_m(t) \geq t\Omega_{M, M_1}(t) \qquad \text{for } t \in [0, 1]; \tag{4.18}$$

here

$$\Omega_{M, M_1}(t) = \frac{1}{t}\int_0^t \left((1-\gamma)\int_x^1 \phi(s)\psi_{M, M_1}(s)\, ds\right)^{1/(1-\gamma)} dx.$$

Now since $\lim_{t \to 0+} \Omega_{M, M_1}(t) = ((1-\gamma)\int_0^1 \phi(s)\psi_{M, M_1}(s)\, ds)^{1/(1-\gamma)}$, Ω_{M, M_1} extends to a continuous function on $[0, 1]$. Consequently there exists a $k_0 > 0$ with $\Omega_{M, M_1}(t) \geq k_0 > 0$ for $t \in [0, 1]$. This together with (4.18) implies

$$y_m(t) \geq k_0 t \qquad \text{for } t \in [0, 1], \tag{4.19}$$

and we have immediately that

$$\begin{aligned} &\{y_m^{(j)}\}_{m \in N_0} \text{ is a bounded, equicontinuous} \\ &\text{family on } [0, 1] \text{ for each } j = 0, 1. \end{aligned} \tag{4.20}$$

Now apply the Arzela–Ascoli Theorem (as in Theorem 3.1) to finish the proof. ∎

Remark 4.1. One could relax condition (4.1) as in Remark 3.1.

EXAMPLE 4.1. Consider the boundary value problem

$$\begin{gathered} y'' + \mu(y')^\beta[y^{-\alpha} + \eta_0 y^a + \eta_1] = 0, \qquad 0 < t < 1 \\ y(0) = y'(1) = 0, \end{gathered} \tag{4.21}$$

with $0 < \alpha < 1$, $0 \leq \beta < 1$, $\eta_0 \geq 0$, $\eta_1 \geq 0$, $a \geq 0$, and $\mu > 0$. If

$$\mu < (2-\beta)\left(\sup_{c\in(0,\infty)} \frac{c}{\left[\eta_0 c^{a+1} + \eta_1 c + c^{1-\alpha}/(1-\alpha)\right]^{1/(2-\beta)}}\right)^{2-\beta}, \tag{4.22}$$

then (4.21) has a solution $y \in C^1[0,1] \cap C^2(0,1)$ with $y > 0$ on $(0,1]$.

We will apply Theorem 4.1 with $\phi = \mu$, $r = u^{\beta}$, $w(u) = u^{-\alpha}$, and $h(u) = \eta_0 u^a + \eta_1$. Clearly, (4.1), (4.2), (4.3), (4.4) (since (4.22) holds), (4.5) (since $0 < \alpha < 1$), (4.6) (with $\psi_{H,L} = H^{-\alpha}$ and $\gamma = \beta$), and (4.7) (since $0 < \alpha < 1$) hold. Theorem 4.1 now establishes the result.

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